An invitation to homological mirror symmetry

Denis Auroux

Harvard University

Uni. Hamburg Kolloquium über Reine Mathematik, April 13, 2021

partially supported by NSF and by the Simons Foundation
(Simons Collaboration on Homological Mirror Symmetry)

Jacobi theta functions and counting triangles

Jacobi theta function on the elliptic curve $E = \mathbb{C} / \mathbb{Z}+\tau\mathbb{Z}$

All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi iz} s(z)$

Only one up to scaling! $\quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z)$.

(Jacobi, 1820s)

$$\exp(\pi i n^2\tau + 2\pi i n(z+\tau)) = \exp(\pi i(n+1)^2\tau + 2\pi i(n+1)z)\,\exp(-\pi i\tau - 2\pi iz)$$

Jacobi theta functions and counting triangles

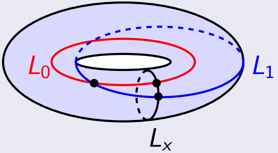
Jacobi theta function on the elliptic curve $E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

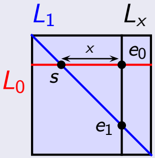
All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z}s(z)$

Only one up to scaling! $s(z) = \vartheta(\tau;z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z)$.

(Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (weighted by area)





Jacobi theta functions and counting triangles

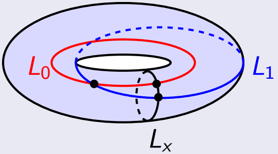
Jacobi theta function on the elliptic curve $E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}$

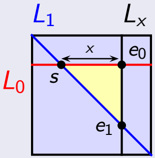
All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z} s(z)$

Only one up to scaling! $\quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$

(Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (weighted by area)





$\boxed{?} = T^{x^2/2}$

Jacobi theta functions and counting triangles

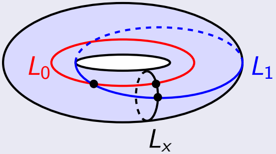
Jacobi theta function on the elliptic curve $E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}$

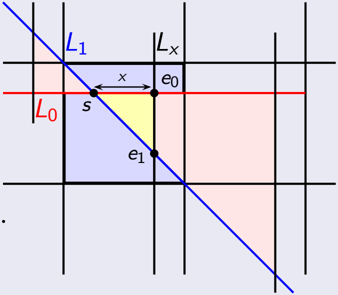
All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z} s(z)$

Only one up to scaling! $\quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$

(Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$





$$\boxed{?} = \cdots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \ldots$$
$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2 + nx}.$$

Jacobi theta functions and counting triangles

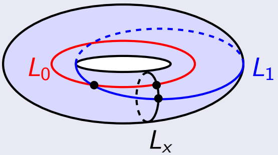
Jacobi theta function on the elliptic curve $E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}$

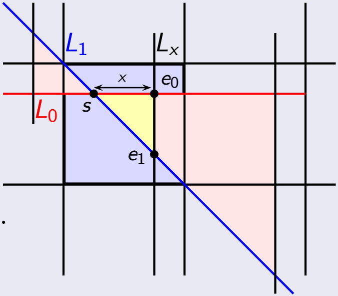
All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z} s(z)$

Only one up to scaling! $\quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$

(Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$





$$\boxed{?} = \cdots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \ldots$$

$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2 + nx} = e^{\pi i \tau x^2} \vartheta(\tau; \tau x)$$

$(T = e^{2\pi i\tau})$

Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E},\mathcal{F}) = Ext^*(\mathcal{E},\mathcal{F})$.

Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0 \ / \sim$
Eg: functions, intersections, cohomology...

Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

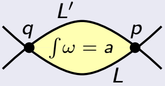
Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E}, \mathcal{F}) = Ext^*(\mathcal{E}, \mathcal{F})$.

Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0$ / $\sim$
Eg: functions, intersections, cohomology...

Symplectic geometry:

(X, ω) loc.$\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, Lagrangian submanifolds L (*dim.* n, $\omega_{|L} = 0$).
Intersections (mod. Hamiltonian isotopy) = Floer cohomology

$CF^*(L, L') = \mathbb{K}^{|L \cap L'|}$
($\otimes$ local coefficients)



$\partial p = T^a q$

Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

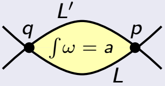
Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E},\mathcal{F}) = Ext^*(\mathcal{E},\mathcal{F})$.

Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0$ / $\sim$
Eg: functions, intersections, cohomology...

Symplectic geometry: **Fukaya category** $\mathcal{F}(X,\omega)$

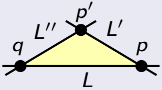
(X,ω) loc.$\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, Lagrangian submanifolds L (*dim.* n, $\omega_{|L} = 0$).
Intersections (mod. Hamiltonian isotopy) = Floer cohomology

$CF^*(L,L') = \mathbb{K}^{|L\cap L'|}$
($\otimes$ local coefficients)



$\partial p = T^a q$

Product $CF(L',L'') \otimes CF(L,L') \to CF(L,L'')$: $\quad p' \cdot p = T^a q$



Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E}, \mathcal{F}) = Ext^*(\mathcal{E}, \mathcal{F})$.

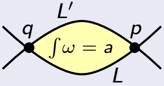
Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0 \;/\; \sim$
Eg: functions, intersections, cohomology...

$\Updownarrow$ **Mirror symmetry:** $D^b Coh(V) \simeq D^\pi \mathcal{F}(X, \omega)$

Symplectic geometry: **Fukaya category** $\mathcal{F}(X, \omega)$

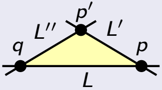
(X, ω) loc.$\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, Lagrangian submanifolds L (*dim.* n, $\omega_{|L} = 0$).
Intersections (mod. Hamiltonian isotopy) = Floer cohomology

$CF^*(L, L') = \mathbb{K}^{|L \cap L'|}$
($\otimes$ local coefficients)



$\partial p = T^a q$

Product $CF(L', L'') \otimes CF(L, L') \to CF(L, L'')$: $\quad p' \cdot p = T^a q$



Example: elliptic curve (Polishchuk-Zaslow)

$E = \mathbb{C} / \mathbb{Z}+\tau\mathbb{Z}, \quad \mathcal{L} = \mathbb{C}^2/(z, v) \sim (z+1, v) \sim (z+\tau, e^{-\pi i\tau - 2\pi i z} v)$

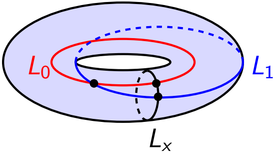
$\dim H^0(E, \mathcal{L}) = 1, \quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$

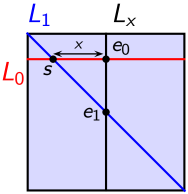
Example: elliptic curve (Polishchuk-Zaslow)

$E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}, \quad \mathcal{L} = \mathbb{C}^2/(z,v) \sim (z+1,v) \sim (z+\tau, e^{-\pi i\tau - 2\pi i z}v)$

$\dim H^0(E,\mathcal{L}) = 1, \quad s(z) = \vartheta(\tau;z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$

$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$



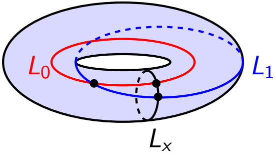


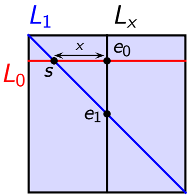
Example: elliptic curve (Polishchuk-Zaslow)

$$E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}, \qquad \mathcal{L} = \mathbb{C}^2/(z,v) \sim (z+1,v) \sim (z+\tau, e^{-\pi i\tau - 2\pi i z} v)$$

$$\dim H^0(E,\mathcal{L}) = 1, \qquad s(z) = \vartheta(\tau;z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$$

$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$





$$L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_x$$

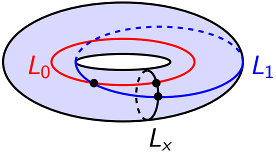
$e_1 \cdot s = \boxed{?}\; e_0 \qquad e_0 \sim$ evaluation $\mathcal{O} \to \mathcal{O}_x$

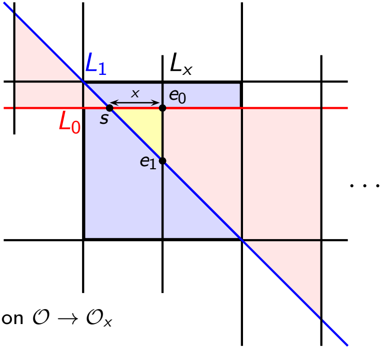
Example: elliptic curve (Polishchuk-Zaslow)

$$E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}, \qquad \mathcal{L} = \mathbb{C}^2/(z,v) \sim (z+1,v) \sim (z+\tau, e^{-\pi i \tau - 2\pi i z} v)$$

$$\dim H^0(E,\mathcal{L}) = 1, \qquad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

$$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$$





$$L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_x$$

$e_1 \cdot s = \boxed{?}\; e_0 \qquad e_0 \sim$ evaluation $\mathcal{O} \to \mathcal{O}_x$

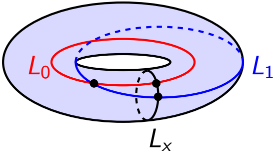
$$\boxed{?} = \dots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \dots$$
$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2 + nx}$$

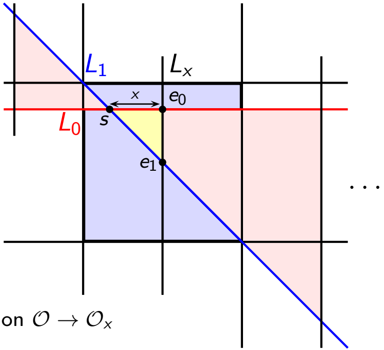
Example: elliptic curve (Polishchuk-Zaslow)

$E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \quad \mathcal{L} = \mathbb{C}^2/(z,v) \sim (z+1,v) \sim (z+\tau, e^{-\pi i\tau - 2\pi i z} v)$

$\dim H^0(E,\mathcal{L}) = 1, \quad s(z) = \vartheta(\tau;z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$

$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$





$$L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_x$$

$e_1 \cdot s = \boxed{?}\ e_0 \qquad e_0 \sim$ evaluation $\mathcal{O} \to \mathcal{O}_x$

$$\boxed{?} = \dots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \dots$$

$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2+nx} = e^{\pi i \tau x^2}\vartheta(\tau;\tau x) \qquad (T = e^{2\pi i\tau})$$

Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith), K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...

Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith), K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...
2. Fano case: $\mathbb{CP}^n$, del Pezzo, toric varieties ... (LG models)
   (Kontsevich, Seidel, Auroux-Katzarkov-Orlov, Abouzaid, FOOO ...)

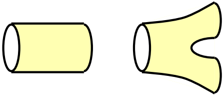
Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith),
   K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...
2. Fano case: $\mathbb{CP}^n$, del Pezzo, toric varieties ... (LG models)
   (Kontsevich, Seidel, Auroux-Katzarkov-Orlov, Abouzaid, FOOO ...)
3. General type case, affine varieties, etc.
   Riemann surfaces, compact (Seidel, Efimov) or non-compact
   (Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Lee, ...)
   hypersurfaces $\subset (\mathbb{C}^*)^n$ or toric varieties (Gammage-Shende, Abouzaid-Auroux, ...)
   ... and beyond

Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith),
   K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...
2. Fano case: $\mathbb{CP}^n$, del Pezzo, toric varieties ... (LG models)
   (Kontsevich, Seidel, Auroux-Katzarkov-Orlov, Abouzaid, FOOO ...)
3. General type case, affine varieties, etc.
   Riemann surfaces, compact (Seidel, Efimov) or non-compact
   (Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Lee, ...)
   hypersurfaces $\subset (\mathbb{C}^*)^n$ or toric varieties (Gammage-Shende, Abouzaid-Auroux, ...)
   ... and beyond

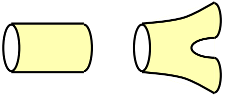
Goal of talk: give a flavor of this program
$\rightsquigarrow$ HMS for all Riemann surfaces starting with



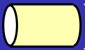
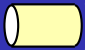
Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith),
   K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...
2. Fano case: $\mathbb{CP}^n$, del Pezzo, toric varieties ... (LG models)
   (Kontsevich, Seidel, Auroux-Katzarkov-Orlov, Abouzaid, FOOO ...)
3. General type case, affine varieties, etc.
   Riemann surfaces, compact (Seidel, Efimov) or non-compact
   (Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Lee, ...)
   hypersurfaces $\subset (\mathbb{C}^*)^n$ or toric varieties (Gammage-Shende, Abouzaid-Auroux, ...)
   ... and beyond

Goal of talk: give a flavor of this program
$\rightsquigarrow$ HMS for all Riemann surfaces starting with

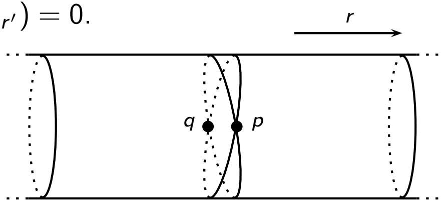


(focusing on HMS itself, ignoring developments from
Strominger-Yau-Zaslow, skeleta, family Floer theory, etc.)

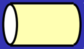
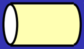
Example 1: $\mathcal{F}_c($  $) \simeq D^b_c($  $)$ (classical)

$X = \mathbb{R} \times S^1$, $\omega = dr \wedge d\theta$, $L_r = \{r\} \times S^1$ (+ local system ξ)

$\Rightarrow HF^*(L_r, L_r) \simeq H^*(S^1, \mathbb{K})$,
$HF^*(L_r, L_{r'}) = 0$.

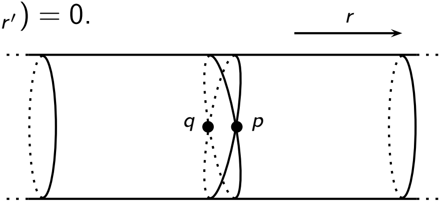


$\partial p = q - q = 0$

Example 1: $\mathcal{F}_c($$) \simeq D^b_c($$)$ (classical)

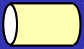
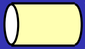
$X = \mathbb{R} \times S^1$, $\omega = dr \wedge d\theta$, $L_r = \{r\} \times S^1$ (+ local system ξ)

$\Rightarrow HF^*(L_r, L_r) \simeq H^*(S^1, \mathbb{K})$,
$HF^*(L_r, L_{r'}) = 0$.



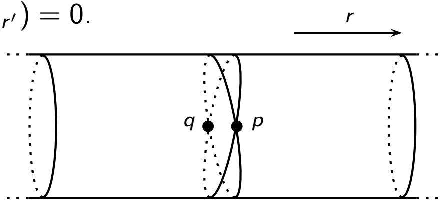
$\partial p = q - q = 0$

- $\mathcal{M}_{pt} = \{(L_r, \xi) \in \mathcal{F}(X)\}/\sim$ has a natural analytic structure
  Coordinate: $z(L_r, \xi) = T^r \mathrm{hol}(\xi) \in \mathbb{K}^*$.
  ($\forall L'$, $CF((L_r, \xi), L')$ has analytic dependence on z)
- $(L_r, \xi) \in \mathcal{F}(X, \omega) \longleftrightarrow \mathcal{O}_z \in D^b(X^\vee = \mathbb{K}^*)$

Example 1: $\mathcal{F}_c($  $) \simeq D^b_c($  $)$ (classical)

$X = \mathbb{R} \times S^1$, $\omega = dr \wedge d\theta$, $L_r = \{r\} \times S^1$ (+ local system ξ)

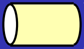
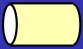
$\Rightarrow HF^*(L_r, L_r) \simeq H^*(S^1, \mathbb{K})$,
$HF^*(L_r, L_{r'}) = 0$.



$\partial p = q - q = 0$

- $\mathcal{M}_{pt} = \{(L_r, \xi) \in \mathcal{F}(X)\}/\sim$ has a natural analytic structure
  Coordinate: $z(L_r, \xi) = T^r \mathrm{hol}(\xi) \in \mathbb{K}^*$.
  ($\forall L'$, $CF((L_r, \xi), L')$ has analytic dependence on z)
- $(L_r, \xi) \in \mathcal{F}(X, \omega) \longleftrightarrow \mathcal{O}_z \in D^b(X^\vee = \mathbb{K}^*)$

Strominger-Yau-Zaslow: X CY, $\pi : X \to B$ Lagrangian torus fibration
$\Rightarrow$ mirror $X^\vee = \{\mathcal{O}_p,\ p \in X^\vee\} = \{(L_b = \pi^{-1}(b), \xi) \in \mathcal{F}(X)\}/\sim$

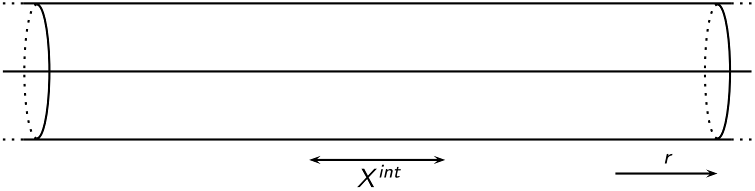
Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b($  $)$

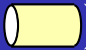
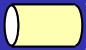
Abouzaid-Seidel
"wrapped Fukaya category"

$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

Hamiltonian perturbation: $H = \frac{1}{2}r^2$, $\quad \phi_H^1(r,\theta) = (r, \theta + r)$.

($\rightarrow$ intersections $\in X^{int}$ + Reeb flow at boundary).



Example 1: $\mathcal{F}_{wr}($$) \simeq D^b($$)$

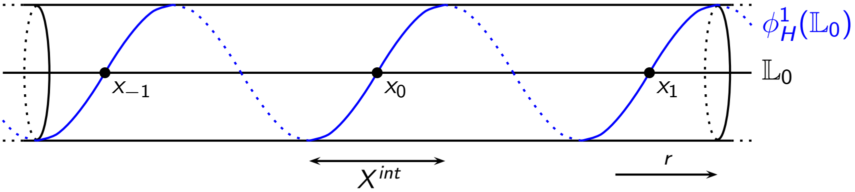
Abouzaid-Seidel
"wrapped Fukaya category"

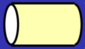
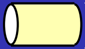
$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

Hamiltonian perturbation: $H = \frac{1}{2}r^2, \quad \phi_H^1(r, \theta) = (r, \theta + r)$.

($\to$ intersections $\in X^{int}$ + Reeb flow at boundary).

$$CW^*(\mathbb{L}_0, \mathbb{L}_0) := CF^*(\phi_H^1(\mathbb{L}_0), \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b($  $)$

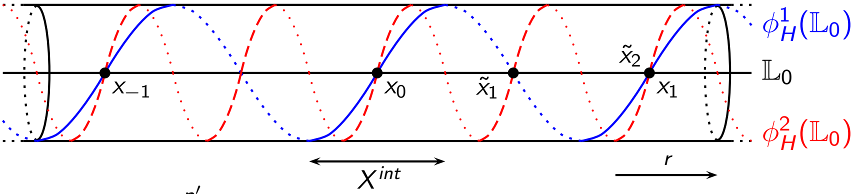
Abouzaid-Seidel
"wrapped Fukaya category"

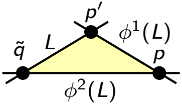
$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

Hamiltonian perturbation: $H = \frac{1}{2} r^2, \quad \phi_H^1(r,\theta) = (r, \theta + r)$.

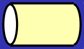
($\rightarrow$ intersections $\in X^{int}$ + Reeb flow at boundary).

$$CW^*(\mathbb{L}_0, \mathbb{L}_0) := CF^*(\phi_H^1(\mathbb{L}_0), \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



Product: 

($\tilde{q} \in \phi^2(L) \cap L \leftrightarrow q \in \phi^1(L) \cap L$ via $r \mapsto 2r$)

Example 1: $\mathcal{F}_{wr}($$) \simeq D^b($$)$

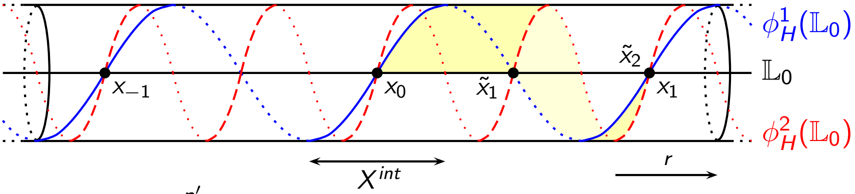
Abouzaid-Seidel
"wrapped Fukaya category"

$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

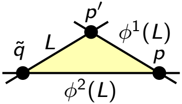
Hamiltonian perturbation: $H = \frac{1}{2} r^2, \quad \phi_H^1(r,\theta) = (r, \theta + r)$.

($\rightarrow$ intersections $\in X^{int}$ + Reeb flow at boundary).

$$CW^*(\mathbb{L}_0, \mathbb{L}_0) := CF^*(\phi_H^1(\mathbb{L}_0), \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$

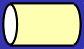


Product:



($\tilde{q} \in \phi^2(L) \cap L \leftrightarrow q \in \phi^1(L) \cap L$ via $r \mapsto 2r$)

$x_k \cdot x_l = x_{k+l}$

Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b($  $)$

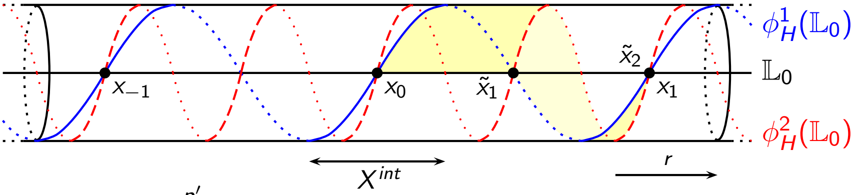
Abouzaid-Seidel
"wrapped Fukaya category"

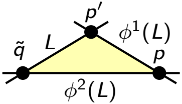
$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

Hamiltonian perturbation: $H = \frac{1}{2} r^2, \quad \phi_H^1(r, \theta) = (r, \theta + r).$

($\to$ intersections $\in X^{int}$ + Reeb flow at boundary).

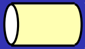
$$CW^*(\mathbb{L}_0, \mathbb{L}_0) := CF^*(\phi_H^1(\mathbb{L}_0), \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



Product: 

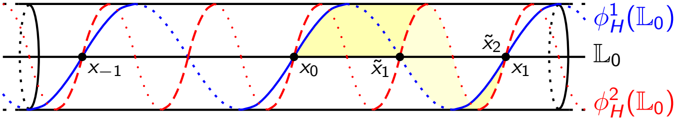
($\tilde{q} \in \phi^2(L) \cap L \leftrightarrow q \in \phi^1(L) \cap L$ via $r \mapsto 2r$)

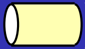
$\boxed{x_k \cdot x_l = x_{k+l}} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}]. \quad (x_k \rightsquigarrow x^k)$

Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b(\mathbb{K}^*)$

Abouzaid-Seidel
"wrapped Fukaya category"

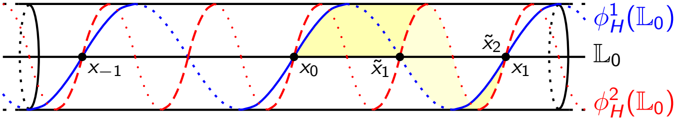
$$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}] \simeq \mathrm{End}(\mathcal{O}_{X^\vee}).$$



Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b(\mathbb{K}^*)$

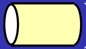
Abouzaid-Seidel
"wrapped Fukaya category"

$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}] \simeq \mathrm{End}(\mathcal{O}_{X^\vee})$.



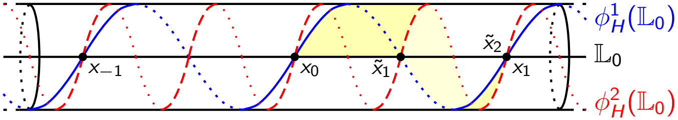
$\mathbb{L}_0$ generates $\mathcal{F}_{wr}(X)$.
Yoneda: $L \mapsto \mathrm{Hom}(\mathbb{L}_0, L)$ gives an embedding $\mathcal{F}_{wr}(X) \hookrightarrow \mathrm{End}(\mathbb{L}_0)$-mod.

Example: $(L_r, \xi) \mapsto HF(\mathbb{L}_0, (L_r, \xi)) \simeq \mathbb{K}[x^{\pm 1}]/(x - z) \quad (z = T^r \mathrm{hol}(\xi))$

Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b(\mathbb{K}^*)$

Abouzaid-Seidel
"wrapped Fukaya category"

$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}] \simeq \mathrm{End}(\mathcal{O}_{X^\vee})$.

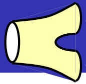


$\mathbb{L}_0$ generates $\mathcal{F}_{wr}(X)$.
Yoneda: $L \mapsto \mathrm{Hom}(\mathbb{L}_0, L)$ gives an embedding $\mathcal{F}_{wr}(X) \hookrightarrow \mathrm{End}(\mathbb{L}_0)$-mod.

Example: $(L_r, \xi) \mapsto HF(\mathbb{L}_0, (L_r, \xi)) \simeq \mathbb{K}[x^{\pm 1}]/(x - z) \quad (z = T^r \mathrm{hol}(\xi))$

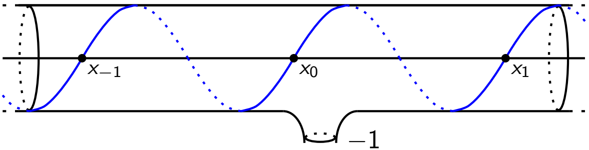
Theorem

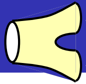
$$\mathcal{F}_{wr}(X) \simeq \mathbb{K}[x^{\pm 1}]\text{-}mod \simeq D^b Coh(X^\vee).$$

Example 2: $\mathcal{F}_{wr}$ () (Abouzaid-A.-Efimov-Katzarkov-Orlov)

$X = S^2 \setminus \{-1, 0, \infty\} = \mathbb{C}^* \setminus \{-1\}$, $\mathbb{L}_0 = \mathbb{R}_+$

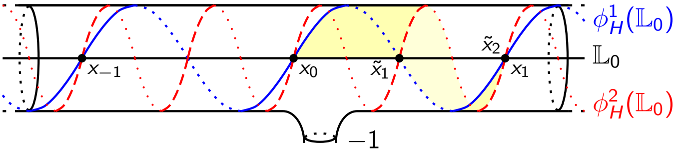
$$\Rightarrow CW(\mathbb{L}_0, \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



Example 2: $\mathcal{F}_{wr}($  $)$ (Abouzaid-A.-Efimov-Katzarkov-Orlov)

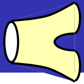
$X = S^2 \setminus \{-1, 0, \infty\} = \mathbb{C}^* \setminus \{-1\}$, $\mathbb{L}_0 = \mathbb{R}_+$

$$\Rightarrow CW(\mathbb{L}_0, \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



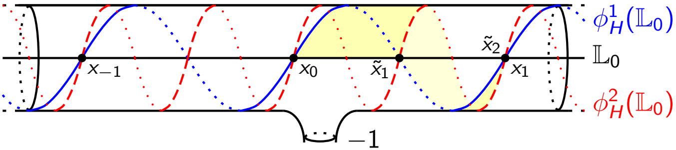
$$x_j \cdot x_i = \begin{cases} x_{i+j} & \text{if } ij \geq 0 \\ 0 & \text{if } ij < 0 \end{cases}$$

$$\Rightarrow \boxed{\mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x, y]/(xy = 0).}$$

Example 2: $\mathcal{F}_{wr}($$)$ (Abouzaid-A.-Efimov-Katzarkov-Orlov)

$X = S^2 \setminus \{-1, 0, \infty\} = \mathbb{C}^* \setminus \{-1\}$, $\mathbb{L}_0 = \mathbb{R}_+$

$$\Rightarrow\ CW(\mathbb{L}_0, \mathbb{L}_0) = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}\, x_i.$$

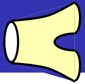


$$x_j \cdot x_i = \begin{cases} x_{i+j} & \text{if } ij \geq 0 \\ 0 & \text{if } ij < 0 \end{cases}$$

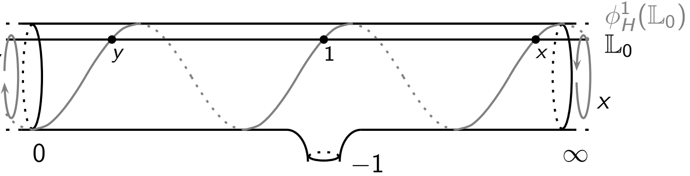
$$\Rightarrow\ \boxed{\mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x, y]/(xy = 0).}$$

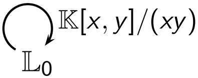
... $X^\vee = \mathrm{Spec}\, \mathbb{K}[x, y]/(xy = 0) = \{xy = 0\} \subset \mathbb{A}^2$?

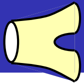
$\mathcal{F}_{wr}(X) \hookrightarrow \mathrm{End}(\mathbb{L}_0)\text{-mod}$??

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

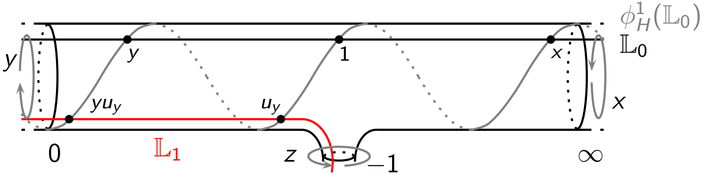
$X = \mathbb{C}^* \setminus \{-1\}$: $\mathbb{L}_0 = (0,\infty)$, $\mathbb{L}_1 = (-1,0)$, $\mathbb{L}_2 = (-\infty,-1)$ generate



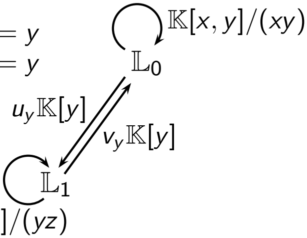


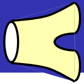
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\}$: $\mathbb{L}_0 = (0,\infty)$, $\mathbb{L}_1 = (-1,0)$, $\mathbb{L}_2 = (-\infty,-1)$ generate

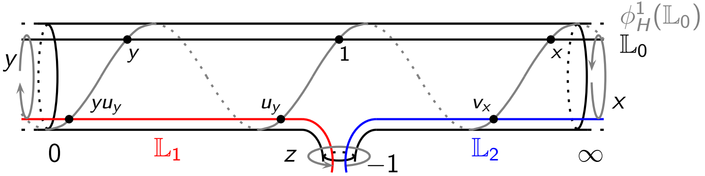


$u_y \cdot v_y = y$
$v_y \cdot u_y = y$

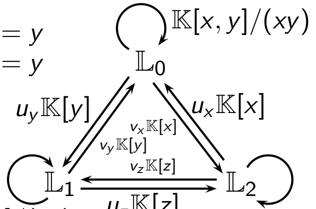


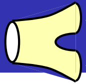
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy = 0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\}$: $\mathbb{L}_0 = (0, \infty)$, $\mathbb{L}_1 = (-1, 0)$, $\mathbb{L}_2 = (-\infty, -1)$ generate

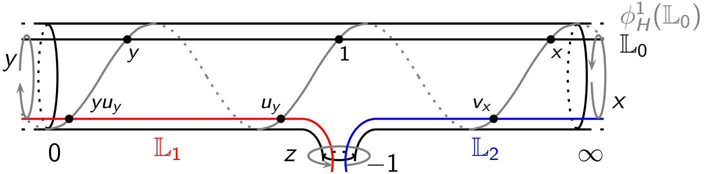


$u_y \cdot v_y = y$
$v_y \cdot u_y = y$

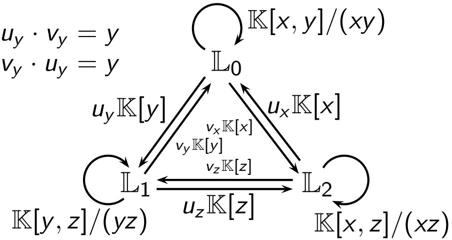


Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\}$: $\mathbb{L}_0 = (0,\infty)$, $\mathbb{L}_1 = (-1,0)$, $\mathbb{L}_2 = (-\infty,-1)$ generate



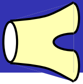
$u_y \cdot v_y = y$
$v_y \cdot u_y = y$



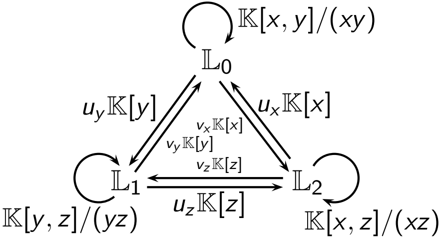
+ exact triangles

$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$

$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

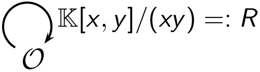
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy = 0\})$ (A-A-E-K-O)

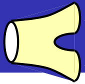
$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$



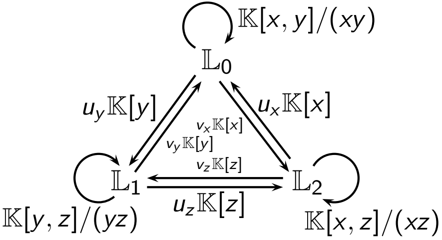
$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$
$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

$X^\vee = \{xy = 0\} = A \cup B \subset \mathbb{A}^2$



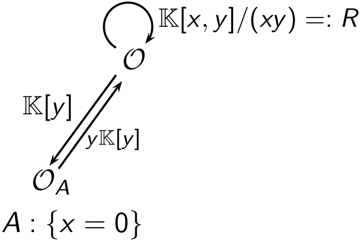
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy = 0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$

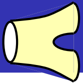


$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$
$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

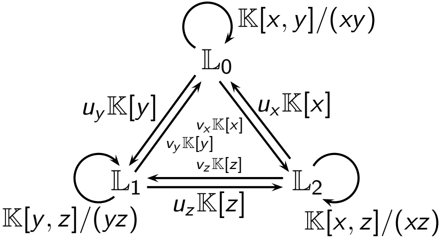
$X^\vee = \{xy = 0\} = A \cup B \subset \mathbb{A}^2$



$\mathrm{Hom}(\mathcal{O}, \mathcal{O}_A) = \mathrm{Hom}_R(R, R/x) \simeq R/x = \mathbb{K}[y]$
$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}) = \mathrm{Hom}_R(R/x, R) \simeq y\mathbb{K}[y]$

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy = 0\})$ (A-A-E-K-O)

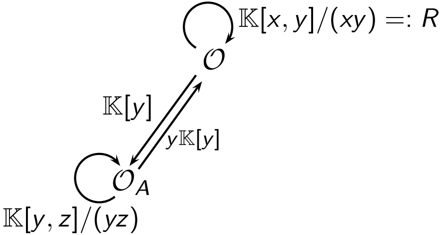
$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$



$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$

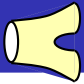
$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$

$X^\vee = \{xy = 0\} = A \cup B \subset \mathbb{A}^2$

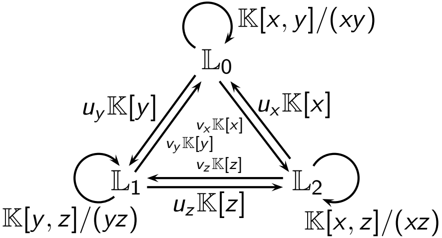


$\mathrm{Hom}(\mathcal{O}, \mathcal{O}_A) = \mathrm{Hom}_R(R, R/x) \simeq R/x = \mathbb{K}[y]$
$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}) = \mathrm{Hom}_R(R/x, R) \simeq y\mathbb{K}[y]$

$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathbb{K}[y]$, $\mathrm{Ext}^{2k}(\mathcal{O}_A, \mathcal{O}_A) \ni z^k$.

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy = 0\})$ (A-A-E-K-O)

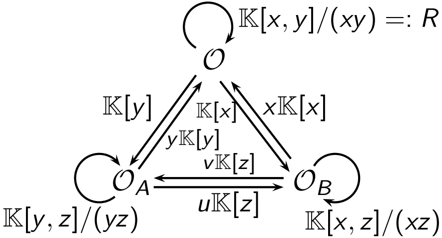
$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$



$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$

$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$

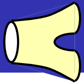
$X^\vee = \{xy = 0\} = A \cup B \subset \mathbb{A}^2$



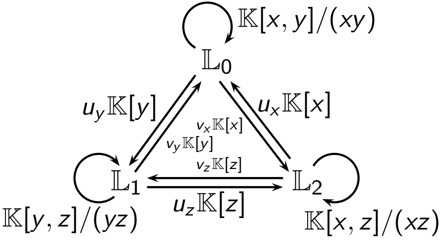
$\mathrm{Hom}(\mathcal{O}, \mathcal{O}_A) = \mathrm{Hom}_R(R, R/x) \simeq R/x = \mathbb{K}[y]$
$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}) = \mathrm{Hom}_R(R/x, R) \simeq y\mathbb{K}[y]$

$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathbb{K}[y]$, $\mathrm{Ext}^{2k}(\mathcal{O}_A, \mathcal{O}_A) \ni z^k$.

same for $\mathcal{O}_B$

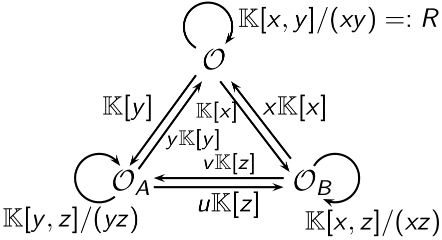
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$



$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$
$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

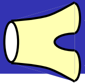
$X^\vee = \{xy=0\} = A \cup B \subset \mathbb{A}^2$



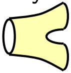
$$\mathcal{O}_B \xrightarrow{x} \mathcal{O} \xrightarrow{1} \mathcal{O}_A \xrightarrow{u} \mathcal{O}_B[1]$$
$$\mathcal{O}_A \xrightarrow{y} \mathcal{O} \xrightarrow{1} \mathcal{O}_B \xrightarrow{v} \mathcal{O}_A[1]$$

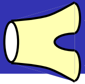
$\Rightarrow$ **Theorem** (A-A-E-K-O)

$$\mathcal{F}_{wr}(X) \simeq D^b Coh(X^\vee)$$

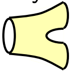
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$ (A-A-E-K-O)

$X = \mathbb{P}^1 \setminus \{-1, 0, \infty\} \longleftrightarrow X^\vee = \{xy = 0\}$:

- $\mathcal{F}_{wr}(X) \simeq D^b Coh(\{xy = 0\})$ lacks symmetry in x, y, z.
- how to extend to higher genus? – gluing  ?

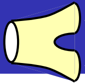
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$ (A-A-E-K-O)

$X = \mathbb{P}^1 \setminus \{-1, 0, \infty\} \longleftrightarrow X^\vee = \{xy = 0\}$:

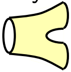
- $\mathcal{F}_{wr}(X) \simeq D^b Coh(\{xy = 0\})$ lacks symmetry in x, y, z.
- how to extend to higher genus? – gluing ?

Stabilization: $X \simeq \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

$(\mathbb{X} = Bl((\mathbb{C}^*)^2 \times \mathbb{C}, X \times 0),\ W = p_{\mathbb{C}}) \longleftrightarrow (\mathbb{X}^\vee = \mathbb{C}^3,\ W^\vee = -xyz)$.

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$ (A-A-E-K-O)

$X = \mathbb{P}^1 \setminus \{-1, 0, \infty\} \longleftrightarrow X^\vee = \{xy = 0\}$:

- $\mathcal{F}_{wr}(X) \simeq D^b Coh(\{xy = 0\})$ lacks symmetry in x, y, z.
- how to extend to higher genus? – gluing ?

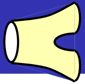
Stabilization: $X \simeq \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

$$(\mathbb{X} = Bl((\mathbb{C}^*)^2 \times \mathbb{C}, X \times 0),\ W = p_{\mathbb{C}}) \longleftrightarrow (\mathbb{X}^\vee = \mathbb{C}^3,\ W^\vee = -xyz).$$

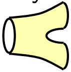
Theorem (A-A-E-K-O)

$\mathcal{F}_{wr}(X) \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee) := D^b Coh(\{xyz = 0\})/Perf.$ *(Orlov)*

$(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2) \longleftrightarrow ([\mathcal{O}_{\{z=0\}}], [\mathcal{O}_{\{x=0\}}], [\mathcal{O}_{\{y=0\}}])$

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$ (A-A-E-K-O)

$X = \mathbb{P}^1 \setminus \{-1, 0, \infty\} \longleftrightarrow X^\vee = \{xy = 0\}$:

- $\mathcal{F}_{wr}(X) \simeq D^b Coh(\{xy = 0\})$ lacks symmetry in x, y, z.
- how to extend to higher genus? – gluing  ?

Stabilization: $X \simeq \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

$(\mathbb{X} = Bl((\mathbb{C}^*)^2 \times \mathbb{C}, X \times 0),\ W = p_{\mathbb{C}}) \longleftrightarrow (\mathbb{X}^\vee = \mathbb{C}^3,\ W^\vee = -xyz)$.

Theorem (A-A-E-K-O)

$\mathcal{F}_{wr}(X) \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee) := D^b Coh(\{xyz = 0\})/Perf$. *(Orlov)*

$(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2) \longleftrightarrow ([\mathcal{O}_{\{z=0\}}], [\mathcal{O}_{\{x=0\}}], [\mathcal{O}_{\{y=0\}}])$

This result extends to all Riemann surfaces (AAEKO, Seidel, Efimov, H. Lee).
Mirror $(\mathbb{X}^\vee, W^\vee)$, $\dim \mathbb{X}^\vee = 3$. (Hori-Vafa, A-A-K)

Geometry of $(\mathbb{X}^\vee, W^\vee)$ (Hori-Vafa, Clarke, Abouzaid-A-Katzarkov, ...)

For an affine plane curve $\Sigma = \{f(x,y)=0\} \subset (\mathbb{C}^*)^2$, mirror:
$\mathbb{X}^\vee$ = toric CY 3-fold determined by *tropicalization* of f,
$W^\vee \in \mathcal{O}(\mathbb{X}^\vee)$, $Z := \{W^\vee = 0\} = \bigcup$ toric strata.
$\mathrm{sing}(Z) = \mathrm{crit}(W^\vee) = \bigcup$ 1-dim. strata = union of $\mathbb{P}^1$ and $\mathbb{A}^1$.

Geometry of $(\mathbb{X}^\vee, W^\vee)$ (Hori-Vafa, Clarke, Abouzaid-A-Katzarkov, ...)

For an affine plane curve $\Sigma = \{f(x,y)=0\} \subset (\mathbb{C}^*)^2$, mirror:
$\mathbb{X}^\vee$ = toric CY 3-fold determined by *tropicalization* of f,
$W^\vee \in \mathcal{O}(\mathbb{X}^\vee)$, $Z := \{W^\vee = 0\} = \bigcup$ toric strata.
$\text{sing}(Z) = \text{crit}(W^\vee) = \bigcup$ 1-dim. strata = union of $\mathbb{P}^1$ and $\mathbb{A}^1$.

Jeff Koons, *Balloon Dog* (photo Librado Romero - The New York Times)

Geometry of $(\mathbb{X}^\vee, W^\vee)$ (Hori-Vafa, Clarke, Abouzaid-A-Katzarkov, ...)

For an affine plane curve $\Sigma = \{f(x,y) = 0\} \subset (\mathbb{C}^*)^2$, mirror:
$\mathbb{X}^\vee$ = toric CY 3-fold determined by *tropicalization* of f,
$W^\vee \in \mathcal{O}(\mathbb{X}^\vee)$, $Z := \{W^\vee = 0\} = \bigcup$ toric strata.
$\mathrm{sing}(Z) = \mathrm{crit}(W^\vee) = \bigcup$ 1-dim. strata = union of $\mathbb{P}^1$ and $\mathbb{A}^1$.

Mirror decompositions: $\Sigma = \bigcup$  $\longleftrightarrow (\mathbb{X}^\vee, W^\vee) = \bigcup (\mathbb{C}^3, -xyz)$

Theorem (Heather Lee)

$\mathcal{F}_{wr}(\Sigma) \simeq \lim \Big\{ \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \rightrightarrows \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \Big\} \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee)$ $(= D^b(Z)/Perf)$

(Related work: Bocklandt, Gammage-Shende, Lekili-Polishchuk, ...)

For an affine plane curve $\Sigma = \{f(x,y)=0\} \subset (\mathbb{C}^*)^2$, mirror:
$\mathbb{X}^\vee$ = toric CY 3-fold determined by *tropicalization* of f,
$W^\vee \in \mathcal{O}(\mathbb{X}^\vee)$, $Z := \{W^\vee = 0\} = \bigcup$ toric strata.
$\mathrm{sing}(Z) = \mathrm{crit}(W^\vee) = \bigcup$ 1-dim. strata = union of $\mathbb{P}^1$ and $\mathbb{A}^1$.

Mirror decompositions: $\Sigma = \bigcup$  $\longleftrightarrow (\mathbb{X}^\vee, W^\vee) = \bigcup (\mathbb{C}^3, -xyz)$

Theorem (Heather Lee)

$\mathcal{F}_{wr}(\Sigma) \simeq \lim \Big\{ \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \rightrightarrows \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \Big\} \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee)$ $(= D^b(Z)/Perf)$

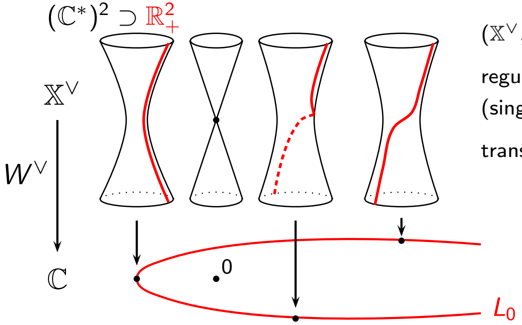
(Related work: Bocklandt, Gammage-Shende, Lekili-Polishchuk, ...)

Theorem (Abouzaid-A.)

The converse also holds! $\mathcal{F}(\mathbb{X}^\vee, W^\vee) \simeq D^b Coh(\Sigma)$

(A.-Efimov-Katzarkov in progress recasts the l.h.s. in terms of $\mathrm{crit}(W^\vee) = \bigcup$ 1-d strata)
(see also C. Cannizzo's thesis for curves in abelian surfaces)
(Abouzaid-A. also holds for X = hypersurface or c.i. in $(\mathbb{C}^*)^n$)

$\mathcal{F}(\mathbb{X}^\vee, W^\vee) \ni L_0, \quad HF(L_0, L_0) \simeq \mathcal{O}(X)$ (Abouzaid-A.)



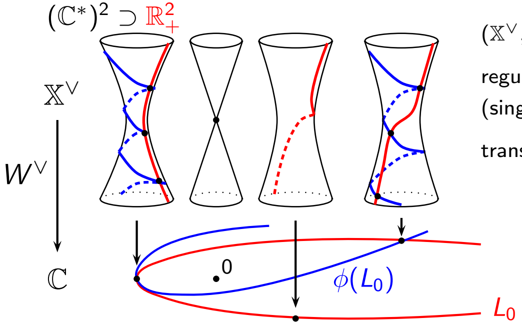
$(\mathbb{X}^\vee, W^\vee)$ mirror to $X = f^{-1}(0) \subset (\mathbb{C}^*)^2$

regular fibers of $W^\vee$ are mirror to $(\mathbb{C}^*)^2$
(sing. fiber $\{W^\vee = 0\}$ = toric degeneration)

transport Lagr. in $(\mathbb{C}^*)^2$ over U-shape
$\leftrightarrow$ restrict sheaf from $(\mathbb{C}^*)^2$ to X

Objects: Lagrangians $L \subset \mathbb{X}^\vee$ s.t. $W^\vee(L) = \text{arc} \to +\infty$.

$\mathcal{F}(\mathbb{X}^\vee, W^\vee) \ni L_0, \quad HF(L_0, L_0) \simeq \mathcal{O}(X)$ (Abouzaid-A.)



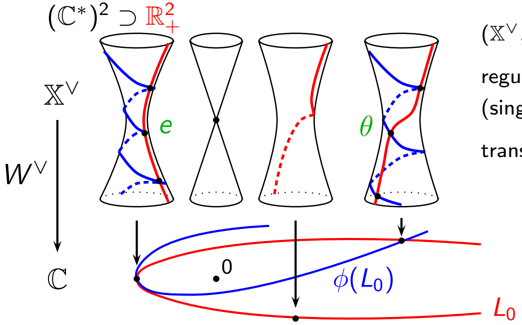
$(\mathbb{X}^\vee, W^\vee)$ mirror to $X = f^{-1}(0) \subset (\mathbb{C}^*)^2$

regular fibers of $W^\vee$ are mirror to $(\mathbb{C}^*)^2$
(sing. fiber $\{W^\vee = 0\}$ = toric degeneration)

transport Lagr. in $(\mathbb{C}^*)^2$ over U-shape
$\leftrightarrow$ restrict sheaf from $(\mathbb{C}^*)^2$ to X

Objects: Lagrangians $L \subset \mathbb{X}^\vee$ s.t. $W^\vee(L) = \text{arc} \to +\infty$. Morphisms: Floer cohomology, with perturbation $H \sim r^2$ along fibers of $W^\vee$ + small rotation of $\mathbb{C}$.

$\mathcal{F}(\mathbb{X}^\vee, W^\vee) \ni L_0, \quad HF(L_0, L_0) \simeq \mathcal{O}(X)$ (Abouzaid-A.)



$(\mathbb{X}^\vee, W^\vee)$ mirror to $X = f^{-1}(0) \subset (\mathbb{C}^*)^2$

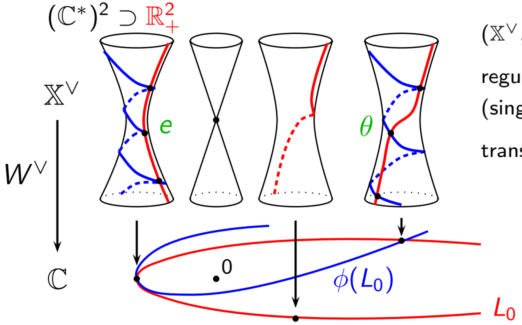
regular fibers of $W^\vee$ are mirror to $(\mathbb{C}^*)^2$
(sing. fiber $\{W^\vee = 0\}$ = toric degeneration)

transport Lagr. in $(\mathbb{C}^*)^2$ over U-shape
$\leftrightarrow$ restrict sheaf from $(\mathbb{C}^*)^2$ to X

Objects: Lagrangians $L \subset \mathbb{X}^\vee$ s.t. $W^\vee(L) = \text{arc} \to +\infty$. Morphisms: Floer cohomology, with perturbation $H \sim r^2$ along fibers of $W^\vee$ + small rotation of $\mathbb{C}$.

$$CF(L_0, L_0) = \mathbb{K}[x^{\pm 1}, y^{\pm 1}]\, e \;\oplus\; \mathbb{K}[x^{\pm 1}, y^{\pm 1}]\, \theta \qquad deg(\theta) = -1.$$

$\mathcal{F}(\mathbb{X}^\vee, W^\vee) \ni L_0, \quad HF(L_0, L_0) \simeq \mathcal{O}(X)$ (Abouzaid-A.)



$(\mathbb{X}^\vee, W^\vee)$ mirror to $X = f^{-1}(0) \subset (\mathbb{C}^*)^2$

regular fibers of $W^\vee$ are mirror to $(\mathbb{C}^*)^2$
(sing. fiber $\{W^\vee = 0\}$ = toric degeneration)

transport Lagr. in $(\mathbb{C}^*)^2$ over U-shape
$\leftrightarrow$ restrict sheaf from $(\mathbb{C}^*)^2$ to X

Objects: Lagrangians $L \subset \mathbb{X}^\vee$ s.t. $W^\vee(L) = \text{arc} \to +\infty$. Morphisms: Floer cohomology, with perturbation $H \sim r^2$ along fibers of $W^\vee$ + small rotation of $\mathbb{C}$.

$$CF(L_0, L_0) = \mathbb{K}[x^{\pm 1}, y^{\pm 1}]\, e \;\oplus\; \mathbb{K}[x^{\pm 1}, y^{\pm 1}]\, \theta \qquad deg(\theta) = -1.$$

$$\partial(x^a y^b \theta) = x^a y^b f(x,y) e \Rightarrow \boxed{HF(L_0, L_0) \simeq \mathbb{K}[x^{\pm 1}, y^{\pm 1}]/(f) \simeq \mathrm{End}(\mathcal{O}_X).}$$

$\mathcal{O}_X$ generates $D^b(X)$; we expect L_0 generates $\mathcal{F}(\mathbb{X}^\vee, W^\vee)$.
($\Rightarrow$ then $\mathcal{F}(\mathbb{X}^\vee, W^\vee) \simeq D^b(X)$...)